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Ladder operators for the associated Laguerre functions

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Abstract

Introducing the associated Laguerre functions in terms of two non-negative integers, we obtain simultaneously and separately realization of the laddering equations with respect to each of the integers by means of two pairs of ladder operators. Besides, two different types of shape-invariance symmetries are realized. This approach leads to a derivation of shape-invariance equations of third type which are realized by two simultaneous raising and lowering operators of two parameters.

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1. Introduction

The associated Laguerre functions that are special cases of confluent hypergeometric functions play an important role in some physical problems including, for example, wavefunctions of hydrogen-like atoms and their bound energies [1]. Moreover, well-known problems of quantum mechanics such as the wave equations of Morse [2] and three-dimensional harmonic oscillator [3] potentials can be converted to the Laguerre differential equation. For instance in [4], Morse and three-dimensional harmonic oscillator solvable models have been derived by using shape-invariance symmetries with respect to n (polynomial degree) and m (dependence parameter) of the associated Laguerre functions, respectively. So, different types of factorizations of the associated Laguerre differential equation into products of ladder operators in the framework of shape-invariance symmetries have attracted much attention in physics problems. Meanwhile by introducing ladder operators corresponding to the degree index of Laguerre polynomials, different coherent states have been obtained for the Morse potential in addition to the supersymmetric structure [5]. From a mathematical point of view, the Laguerre polynomials are studied in connection with generating functions [6], closed-form sums [7] and arbitrary fractional orders [8]. In [9], by using the recursion relations with respect to the polynomial degree, a pair of ladder operators has been deduced for some orthogonal polynomials such as Laguerre's. Cotfas [10] showed the existence of ladder operators for the hypergeometric-type functions so that they realize simultaneously shape-invariance symmetries with respect to two different parameters. We followed Cotfas' idea for the associated hypergeometric functions [11]. In this manuscript we study simultaneous shape-invariance symmetry with respect to more than one parameter for the associated Laguerre functions. It is shown that the obtained results are somewhat different from the associated hypergeometric functions case. The results of this manuscript may be used for investigating the superalgebras of the Morse and three-dimensional harmonic oscillator potentials [12], as well as quantum splitting and the coherent states of the Landau problem [13, 14] (motion of a charged and spinless particle on a flat surface in the presence of a uniform magnetic field along the z-axis). Meanwhile, the ladder operators corresponding to the simultaneous shift of two indices of the associated Laguerre functions can be applied to the investigation of supersymmetric structures for the radial bound states of hydrogen-like atoms, and also some other problems.

In this manuscript by introducing the associated Laguerre functions (confluent hypergeometric functions as finite series) in terms of two parameters n and m which describe the polynomial degree and dependence index, respectively, we realize a square integrability condition and their orthogonality with the same m but different n, with respect to an inner product with the weight function $x^{\alpha}\,\mathrm{e}^{-\beta x}$ in the interval $x \in (0, \infty)$. Then by applying this integrability condition, we factorize the associated Laguerre differential equation into a product of first-order differential operators in three different ways as shape-invariance equations. These shape-invariance relations are realized by ladder operators shifting only n, shifting only m and shifting indices n and m simultaneously and agreeably. In contrast to the associated hypergeometric functions, it is shown that the fourth type of shape invariance for the Laguerre's ones, which is expected to shift the indices n and m simultaneously and inversely by the first-order differential ladder operators, does not exist.

2. Shape-invariance equations with respect to *n* and *m*

Let us firstly consider a linear second-order differential operator with given real parameters $\alpha > -1$ and $\beta > 0$ as

$$\mathcal{L}^{(\alpha,\beta)}(x) := x^{-\alpha}\,\mathrm{e}^{\beta x}\,\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\alpha+1}\,\mathrm{e}^{-\beta x}\,\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{1}$$

Lemma 1. *The operator $\mathcal{L}^{(\alpha,\beta)}(x)$ has the following properties:*

(a) It is a self-adjoint operator with respect to an inner product with the weight function $x^{\alpha}\,\mathrm{e}^{-\beta x}$ in the interval $x \in (0, \infty)$.

(b) The action of the operator $\mathcal{L}^{(\alpha,\beta)}(x)$ on any polynomial of arbitrary degree is such that the degree of the polynomial is not increased.

(c) If we show the eigenfunctions of the operator $\mathcal{L}^{(\alpha,\beta)}(x)$ with $L_n^{(\alpha,\beta)}(x)$ as a polynomial exactly of degree n, then we can conclude its eigenvalue equation as follows:

$$xL_n^{\prime\prime(\alpha,\beta)}(x) + (1+\alpha-\beta x)\,L_n^{\prime(\alpha,\beta)}(x) + n\beta L_n^{(\alpha,\beta)}(x) = 0 \qquad n = 0, 1, 2, \ldots. \tag{2}$$

Proof. The proof is straightforward. □

Equation (2) is the differential equation corresponding to the Laguerre orthogonal polynomials of arbitrary degree n [15].

Lemma 2. *The orthogonal Laguerre polynomials as particular solutions of the differential equation (2) have as a representation the so-called Rodrigues formula:*

$$L_n^{(\alpha,\beta)}(x) = \frac{a_n(\alpha,\beta)}{x^\alpha \,\mathrm{e}^{-\beta x}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^{\alpha+n}\,\mathrm{e}^{-\beta x}) \tag{3}$$

where $a_n(\alpha,\beta)$ are the normalization coefficients.

Proof. A complete proof can be seen in [15]. □

It is easily seen that the coefficient of the highest power of x, x^n, for $L_n^{(\alpha,\beta)}(x)$ is

$$L_n^{(\alpha,\beta)}(x) = a_n(\alpha,\beta)\,(-\beta)^n\, x^n + O(x^{n-1}). \tag{4}$$

Therefore, considering the confluent hypergeometric function as a finite series

$${}_1F_1(-n;\alpha+1;\beta x) = \Sigma_{k=0}^n \frac{(-n)_k}{(\alpha+1)_k}\frac{(\beta x)^k}{k!} \tag{5}$$

in which $(a)_k$ denotes the shifted factorial (or Pochhammer symbol): $(a)_k = a(a+1)\cdots(a+k-1)$ with $k > 0$, and $(a)_0 = 1$, the Laguerre polynomials $L_n^{(\alpha,\beta)}(x)$ can be expressed as follows:

$$L_n^{(\alpha,\beta)}(x) = a_n(\alpha,\beta)\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}\,{}_1F_1(-n;\alpha+1;\beta x). \tag{6}$$

Lemma 3. *The inner product of the orthogonal Laguerre polynomials with respect to the weight function $x^\alpha\,\mathrm{e}^{-\beta x}$ in the interval $x \in (0,\infty)$ is computed as follows:*

$$\int_0^\infty L_n^{(\alpha,\beta)}(x) L_{n'}^{(\alpha,\beta)}(x) x^\alpha\,\mathrm{e}^{-\beta x}\,\mathrm{d}x = \delta_{nn'} h_n^2(\alpha,\beta) \qquad n,n' \in \{0,1,2,\ldots\} \tag{7}$$

where

$$h_n^2(\alpha,\beta) = \frac{\Gamma(n+1)\Gamma(\alpha+n+1)}{\beta^{\alpha+1}} a_n^2(\alpha,\beta). \tag{8}$$

Proof. This follows immediately from integration by parts. □

Lemma 4. *We have the following associated Laguerre functions differential equation,*

$$x L''^{(\alpha,\beta)}_{n,m}(x) + (1+\alpha-\beta x) L'^{(\alpha,\beta)}_{n,m}(x) + \left[\left(n-\frac{m}{2}\right)\beta - \frac{m}{2}\left(\alpha+\frac{m}{2}\right)\frac{1}{x}\right] L^{(\alpha,\beta)}_{n,m}(x) = 0$$
$$0 \leqslant m \leqslant n < +\infty \tag{9}$$

with the solutions as

$$L_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha,\beta)}{x^{\alpha+\frac{m}{2}}\,\mathrm{e}^{-\beta x}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} (x^{\alpha+n}\,\mathrm{e}^{-\beta x}) \tag{10}$$

where $a_{n,m}(\alpha,\beta)$ are the normalization coefficients.

Proof. By differentiating the differential equation (2) m times we obtain a new differential equation similar to (2), but with new parameters $\alpha+m$ and $n-m$ instead of α and n, respectively. Thus for the obtained differential equation, we have a polynomial solution of degree $n-m$ as $L_{n-m}^{(\alpha+m,\beta)}(x)$. Finally, it is easily seen that the associated Laguerre functions

$$L_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha,\beta)}{a_{n-m}(\alpha+m,\beta)} x^{\frac{m}{2}} L_{n-m}^{(\alpha+m,\beta)}(x) \tag{11}$$

satisfy the differential equation (9). □

It is evident that by choosing $m = 0$, the associated Laguerre differential equation (9) is reduced to the differential equation (2) for the Laguerre polynomials. Note that the associated Laguerre function $L_{n,m}^{(\alpha,\beta)}(x)$ will be a polynomial in terms of integer or half-integer powers of x if m is chosen as an even or odd integer, respectively. Clearly for an odd m, the associated Laguerre function $L_{n,m}^{\alpha,\beta}(x)$ is a polynomial in terms of powers of x apart from the coefficient $\sqrt{x}$.

Lemma 5. *We have*

$$\int_0^\infty L_{n,m}^{(\alpha,\beta)}(x) L_{n',m}^{(\alpha,\beta)}(x) x^\alpha \, \mathrm{e}^{-\beta x} \, \mathrm{d}x = \delta_{nn'} h_{n,m}^2(\alpha,\beta) \qquad 0 \leqslant m \leqslant \min\{n,n'\} \tag{12}$$

where

$$h_{n,m}^2(\alpha,\beta) = \frac{\Gamma(n-m+1)\Gamma(\alpha+n+1)}{\beta^{\alpha+m+1}} a_{n,m}^2(\alpha,\beta). \tag{13}$$

Proof. The proof follows by using lemma 3 and the formula (11). □

We shall determine the normalization coefficients by realizing the laddering equations. Before investigating the laddering equations, we obtain the shape-invariance symmetry equations with respect to two parameters n and m.

Proposition 1. *The associated Laguerre functions differential equation (9) is factorized into a product of first-order differential operators as*

(a) shape-invariance symmetry equations (of first type) with respect to n, i.e. as equations (n,m) and $(n-1,m)$

$$\begin{aligned} A_+(n,m;x)A_-(n,m;x)L_{n,m}^{(\alpha,\beta)}(x) &= E(n,m)L_{n,m}^{(\alpha,\beta)}(x) \\ A_-(n,m;x)A_+(n,m;x)L_{n-1,m}^{(\alpha,\beta)}(x) &= E(n,m)L_{n-1,m}^{(\alpha,\beta)}(x) \end{aligned} \tag{14}$$

with

$$\begin{aligned} A_+(n,m;x) &= x\frac{\mathrm{d}}{\mathrm{d}x} - \beta x + \frac{1}{2}(2\alpha+2n-m) \\ A_-(n,m;x) &= -x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}(2n-m) \end{aligned} \tag{15}$$

$$E(n,m) = (n-m)(n+\alpha). \tag{16}$$

(b) shape-invariance symmetry equations (of second type) with respect to m, i.e. as equations (n,m) and $(n,m-1)$

$$\begin{aligned} A_+(m;x)A_-(m;x)L_{n,m}^{(\alpha,\beta)}(x) &= \mathcal{E}(n,m)L_{n,m}^{(\alpha,\beta)}(x) \\ A_-(m;x)A_+(m;x)L_{n,m-1}^{(\alpha,\beta)}(x) &= \mathcal{E}(n,m)L_{n,m-1}^{(\alpha,\beta)}(x) \end{aligned} \tag{17}$$

with

$$\begin{aligned} A_+(m;x) &= \sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{m-1}{2\sqrt{x}} \\ A_-(m;x) &= -\sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2\alpha+m-2\beta x}{2\sqrt{x}} \end{aligned} \tag{18}$$

$$\mathcal{E}(n,m) = (n-m+1)\beta. \tag{19}$$

Proof. The proof can be made by means of a direct substitution of the explicit forms of $A_{\pm}(n, m; x)$, $E(n, m)$, $A_{\pm}(m; x)$ and $\mathcal{E}(n, m)$ in equations (14) and (17), and converting them to the differential equation (9). □

Note that the operators $A_+(m; x)$ and $A_-(m; x)$ ($A_+(n, m; x)$ and $A_-(n, m; x)$) are (not) adjoint to each other with respect to an inner product with the weight function $x^\alpha \, \mathrm{e}^{-\beta x}$ in the interval $x \in (0, \infty)$.

3. Simultaneous realization of laddering equations with respect to *n* and *m*

Now by using the shape-invariance symmetry equations (14) and (17), we can obtain the raising and lowering relations of the indices n and m of the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(x)$. Clearly, the realization of the shape-invariance symmetry equations (14) and (17) does not impose any constraint on the normalization coefficients $a_{n,m}(\alpha, \beta)$. However, the realization of the laddering equations with respect to n and m imposes separately recursion relations on the normalization coefficients with respect to n and m, respectively. These recursion relations determine the fact that now function $a_{n,m}(\alpha, \beta)$ is from n and m.

Proposition 2. *For a given m, the raising and lowering relations of the index n,*

$$A_+(n, m; x) L_{n-1,m}^{(\alpha,\beta)}(x) = \sqrt{E(n, m)} L_{n,m}^{(\alpha,\beta)}(x) \tag{20a}$$

$$A_-(n, m; x) L_{n,m}^{(\alpha,\beta)}(x) = \sqrt{E(n, m)} L_{n-1,m}^{(\alpha,\beta)}(x) \tag{20b}$$

and for a given n, the raising and lowering relations of the index m,

$$A_+(m; x) L_{n,m-1}^{(\alpha,\beta)}(x) = \sqrt{\mathcal{E}(n, m)} L_{n,m}^{(\alpha,\beta)}(x) \tag{21a}$$

$$A_-(m; x) L_{n,m}^{(\alpha,\beta)}(x) = \sqrt{\mathcal{E}(n, m)} L_{n,m-1}^{(\alpha,\beta)}(x) \tag{21b}$$

are simultaneously established if the normalization coefficient $a_{n,m}(\alpha, \beta)$ is chosen as

$$a_{n,m}(\alpha, \beta) = (-1)^m \sqrt{\frac{\beta^m}{\Gamma(n - m + 1)\Gamma(\alpha + n + 1)}} C(\alpha, \beta) \qquad 0 \leqslant m \leqslant n < +\infty \tag{22}$$

where $C(\alpha, \beta)$ is an arbitrary real constant independent of n and m. Therefore, $A_{\pm}(n, m; x)$ and $A_{\pm}(m; x)$ are the ladder operators on the indices n and m of the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(x)$, respectively.

Proof. Using equation (10) in (20*a*) and applying equation (4) as well, one may compare the coefficients of the highest power of x, $x^{n-\frac{m}{2}}$ on both sides, then the following recursion relation with respect to the index n is obtained:

$$a_{n,m}(\alpha, \beta) = \frac{a_{n-1,m}(\alpha, \beta)}{\sqrt{(n - m)(\alpha + n)}} \qquad 0 \leqslant m < n < +\infty. \tag{23}$$

If we follow a similar procedure in connection with equation (20*b*), then we will find that the coefficient of the highest power of x, that is $x^{n-\frac{m}{2}}$, is identically zero on both sides. Repeated application of the recursion relation (23) results in

$$a_{n,m}(\alpha, \beta) = \sqrt{\frac{\Gamma(\alpha + m + 1)}{\Gamma(n - m + 1)\Gamma(\alpha + n + 1)}} a_{m,m}(\alpha, \beta) \qquad 0 \leqslant m \leqslant n < +\infty. \tag{24}$$

Moreover, using equation (10) in each of equations (21*a*) and (21*b*) then applying (4), one may obtain the following recursion relation on the index *m* by means of comparing the coefficients of the highest power of x, $x^{n-\frac{m}{2}}$ and $x^{n-\frac{m}{2}+\frac{1}{2}}$ respectively, on both sides of them:

$$a_{n,m}(\alpha,\beta) = \frac{a_{n,m+1}(\alpha,\beta)}{-\sqrt{(n-m)\beta}} \qquad 0 \leqslant m < n < +\infty. \tag{25}$$

The recursion relation (25) immediately gives

$$a_{n,m}(\alpha,\beta) = \frac{a_{n,n}(\alpha,\beta)}{(-1)^{n-m}\beta^{\frac{n-m}{2}}\sqrt{\Gamma(n-m+1)}} \qquad 0 \leqslant m \leqslant n < +\infty. \tag{26}$$

Comparing the results (24) and (26), it appears that

$$a_{n,n}(\alpha,\beta) = \frac{(-1)^n\beta^{\frac{n}{2}}}{\sqrt{\Gamma(\alpha+n+1)}}C(\alpha,\beta) \qquad C(\alpha,\beta) = \sqrt{\Gamma(\alpha+1)}a_{0,0}(\alpha,\beta) \tag{27}$$

where $n = 0, 1, 2, \ldots$. Certainly, a relation similar to (27) is satisfied when $n = m$. Therefore by using each of equations (24) and (26), the relation (22) for the normalization coefficients is obtained. Although in deriving the relation (22) we have not used (20*b*) however, one may consider the realization of the relation (20*b*) by means of using (22). This completes the proof. □

It is important to note that in (22) we have essentially derived the normalization coefficients in terms of *n* and *m* in such a way that the associated Laguerre functions satisfy simultaneously the laddering relations with respect to both indices *n* and *m*. Clearly, the laddering relations (20) for a given *m*, and (21) for a given *n* are infinite and finite respectively, since $m \leqslant n < +\infty$. The norm of the associated Laguerre functions is determined by fixing the normalization coefficients as (22).

Corollary 1. *The norm of the associated Laguerre functions* $L_{n,m}^{(\alpha,\beta)}(x)$ *is independent of n and m, and is given by*

$$h_{n,m}^2(\alpha,\beta) = \frac{C^2(\alpha,\beta)}{\beta^{\alpha+1}}. \tag{28}$$

Proof. It follows immediately by substituting (22) in (13). □

It must be mentioned that the norm of the Laguerre functions as (28) that has been determined due to simultaneous realization of laddering equations with respect to *n* and *m* is neither a function of *n* nor, in contrast to the norm of the associated hypergeometric functions [11], *m*. In other words, all of the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(x)$ for all *n* and *m* have the same norm.

Corollary 2. *There are the following two algebraic solutions for the associated Laguerre functions differential equation:*

$$L_{n,m}^{(\alpha,\beta)}(x) = \sqrt{\frac{\Gamma(\alpha+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+n+1)}}A_+(n,m;x)A_+(n-1,m;x)\cdots A_+(m+1,m;x)L_{m,m}^{(\alpha,\beta)}(x) \qquad n \geqslant m+1 \tag{29}$$

$$L_{n,m}^{(\alpha,\beta)}(x) = \frac{A_-(m+1;x)A_-(m+2;x)\cdots A_-(n;x)L_{n,n}^{(\alpha,\beta)}(x)}{\sqrt{\beta^{n-m}\Gamma(n-m+1)}} \qquad m \leqslant n-1 \tag{30}$$

where

$$L_{m,m}^{(\alpha,\beta)}(x) = a_{m,m}(\alpha,\beta)x^{\frac{m}{2}}. \tag{31}$$

Proof. Considering $E(m, m) = \mathcal{E}(n, n + 1) = 0$, the first-order differential equations $A_-(m, m; x)L_{m,m}^{(\alpha,\beta)}(x) = 0$ and $A_+(n + 1; x)L_{n,n}^{(\alpha,\beta)}(x) = 0$ are obtained from equations (20*b*) and (21*a*), respectively. The solution of the first differential equation is (31) which is also the solution of the second differential equation if m is replaced by n. For given m and n, using the laddering relations (20*a*) and (21*b*) one may obtain the algebraic solutions (29) and (30) for the associated Laguerre functions, respectively. □

Note that the algebraic solution (31) is consistent with the analytic solution (10).

4. Shape-invariance and laddering equations with respect to *n* and *m* simultaneously

The laddering equations (20) and (21), which shift n and m respectively, lead to the derivation of a new type of factorization for the differential equation (9) as the shape-invariance symmetry equations with the indices (n, m) and $(n - 1, m - 1)$. This factorization is realized by a pair of the ladder operators whose corresponding laddering equations shift both of the indices n and m simultaneously and agreeably.

Proposition 3. *Let us define two new ladder operators as*

$$\begin{aligned} A_{+,+}(m; x) &:= A_+(m; x)A_+(n, m - 1; x) - A_+(n, m; x)A_+(m; x) \\ A_{-,-}(m; x) &:= A_-(n, m - 1; x)A_-(m; x) - A_-(m; x)A_-(n, m; x). \end{aligned} \tag{32}$$

(a) They satisfy the raising and lowering relations with respect to n and m, simultaneously as

$$A_{+,+}(m; x)L_{n-1,m-1}^{(\alpha,\beta)}(x) = \sqrt{(\alpha + n)\beta}L_{n,m}^{(\alpha,\beta)}(x) \tag{33a}$$

$$A_{-,-}(m; x)L_{n,m}^{(\alpha,\beta)}(x) = \sqrt{(\alpha + n)\beta}L_{n-1,m-1}^{(\alpha,\beta)}(x). \tag{33b}$$

So, the operator $A_{+,+}(n, m; x)$ increases both of the indices n and m but the operator $A_{-,-}(n, m; x)$ decreases both of them.

(b) They satisfy shape-invariance symmetry equations (of the third type) with respect to the indices n and m as equations (n, m) and $(n - 1, m - 1)$:

$$\begin{aligned} A_{+,+}(m; x)A_{-,-}(m; x)L_{n,m}^{(\alpha,\beta)}(x) &= (\alpha + n)\beta L_{n,m}^{(\alpha,\beta)}(x) \\ A_{-,-}(m; x)A_{+,+}(m; x)L_{n-1,m-1}^{(\alpha,\beta)}(x) &= (\alpha + n)\beta L_{n-1,m-1}^{(\alpha,\beta)}(x). \end{aligned} \tag{34}$$

(c) They are first-order differential operators with the following explicit forms:

$$\begin{aligned} A_{+,+}(m; x) &= \sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{m - 1 + 2\beta x}{2\sqrt{x}} = A_+(m; x) - \beta\sqrt{x} \\ A_{-,-}(m; x) &= -\sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2\alpha + m}{2\sqrt{x}} = A_-(m; x) - \beta\sqrt{x}. \end{aligned} \tag{35}$$

Proof. The laddering relations (33) are proved by applying the laddering relations (20) and (21) in the definitions (32). The shape-invariance symmetry equations (34) are established by the laddering relations (33). The explicit forms of the differential operators $A_{\pm,\pm}(m; x)$ are obtained by the explicit forms of the differential operators $A_\pm(n, m; x)$ and $A_\pm(m; x)$ which are given by equations (15) and (18), respectively. □

It is noted that simultaneous raising and lowering operators of both indices n and m of the associated Laguerre functions are only functions of m. However, the operators corresponding to the associated hypergeometric functions are functions of n and m in general. Note that

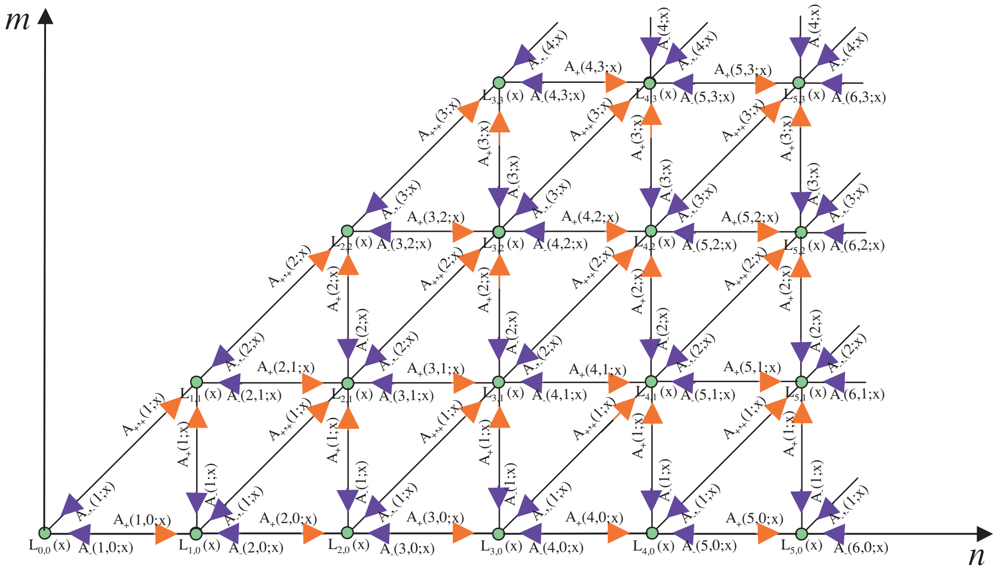

Figure 1. The plane of displacements of the associated Laguerre functions in three different ways by the ladder operators shifting only n, shifting only m, shifting indices n and m simultaneously and agreeably.

(This figure is in colour only in the electronic version)

each of equations (14), (17) and (34) is converted to the differential equation (9) after some manipulation. In fact, they are different types of factorizations of (9) as the shape-invariance symmetry equations. Meanwhile by using each of the laddering relations (20), (21) and (33), one may obtain a pair of recursion relations on three associated Laguerre functions with respect to n and m. In figure 1, we have shown all the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(x)$ as points (n, m) with the restriction $0 \leqslant m < n < \infty$ in the flat plane with n and m as the horizontal and vertical axes, respectively. The ladder operators $A_+(n, m; x)$ and $A_-(n, m; x)$, $A_+(m, x)$ and $A_-(m; x)$, $A_{+,+}(m, x)$ and $A_{-,-}(m; x)$ displace the associated Laguerre functions laid on the lines of horizontal, vertical, parallel with the bisector of the first quadrant to the right and left sides, up and down, right upper corner and left lower corner, respectively.

Remark. Using the explicit forms of the differential operators given in equations (15), (18) and (35), it is easily shown that the following relations are established:

$$\begin{aligned} A_{+,-}(m; x) &:= A_+(n, m-1; x)A_-(m; x) - A_-(m; x)A_+(n, m; x) = 0 \\ A_{-,+}(m; x) &:= A_+(m; x)A_-(n, m-1; x) - A_-(n, m; x)A_+(m; x) = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} A_{+,2+}(m; x) &:= A_+(m; x)A_{+,+}(m-1; x) - A_{+,+}(m; x)A_+(m-1; x) = 0 \\ A_{-,2-}(m; x) &:= A_-(m-1; x)A_{-,-}(m; x) - A_{-,-}(m-1; x)A_-(m; x) = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} A_{2+,+}(m; x) &:= A_+(n, m; x)A_{+,+}(m; x) - A_{+,+}(m; x)A_+(n-1, m-1; x) = 0 \\ A_{2-,-}(m; x) &:= A_-(n-1, m-1; x)A_{-,-}(m; x) - A_{-,-}(m; x)A_-(n, m; x) = 0. \end{aligned} \tag{38}$$

These relations can also be obtained by using the laddering relations (20), (21) and (33) via the action of the operators on the space of associated Laguerre functions. The relations (36) show that, in contrast to the associated hypergeometric functions case [11], it is not possible to obtain first-order differential operators so that they increase one of the indices and decrease the other one simultaneously. Furthermore, the relations (37) and (38) indicate that first-order ladder differential operators which increase (decrease) one of the indices by one unit and the other one by two units, do not exist. In fact, the relations (37) and (38) give rise to realization of commutation relations corresponding to two generator bunches of Lie algebra h_4 as the relations (13) in [14]. The mentioned fact provides realization of Lie algebras $su(2)$ and $su(1, 1)$ by two generator bunches which have been introduced there.

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